Rational Numbers: Part 1

Figure 1: CartoonStock ©

Package Summary

- The Base – 10 Number System
  - Reviewing Long Division
  - Understanding Fractions
  - Equivalent Fractions
- Examples of Equivalent Fractions
- Improper Fractions and Mixed Numbers
  - Ratios and Proportions
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1. **numeration system**: A system that uses symbols or **numerals**, to represent numbers. e.g. Tally Numeration System, Roman Numeration System, or the Babylonian Numeration System.

2. **place value**: Place values are used by some numeration systems to give a specific numerical value to a digit/symbol, depending on its position.

3. **Hindu-Arabic Numeration System (Base – 10 system)**: The **Base – 10 system** is the number system that we presently use, and has the following characteristics:
   - The digits used include: 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9.
   - Numbers are constructed by grouping sets of 10 so that the place values are based on **powers of 10**. When reading from **right to left**, if a digit has the first place value, it represents a group of less than ten ones. If a digit has the second place value, it represents groups of $10 = 10^1$ ones. If a digit has the third place value, it represents groups of $100 = 10^2$ ones, and so on.
   - **Place values** for integers include: ones place, tens place, hundreds place, thousands place, ten-thousands place, etc. (when reading digits from **right to left**). For example, the underlined digit in 38,157 is in the __________ place.

![Place Values Diagram](image)

4. **expanded form of a number**: The expanded form of a number expresses the number as a sum of its digits multiplied by their respective place values. For example,

\[
245,036 = 2(10^5) + 4(10^4) + 5(10^3) + 0(10^2) + 3(10^1) + 6(1)
\]
\[
= 200,000 + 40,000 + 5,000 + 30 + 6
\]

Any partially expanded forms will be referred to as **semi-expanded**.
Understanding Long Division

Consider the following example: Evaluate $2683 \div 3$.

Solution: Let’s begin by considering a semi-expanded form of 2683. Then, using this form, we will systematically determine how many multiples of 3 we have:

\[
2683 = 2600 + 83 \\
= (2400 + 200) + 83 \\
= 3(800) + 283 \\
= 3(800) + 280 + 3 \\
= 3(800) + (270 + 10) + 3 \\
= 3(800) + 3(90) + 13 \\
= 3(800) + 3(90) + (12 + 1) \\
= 3(800) + 3(90) + 3(4) + 1 \\
= 3(894) + 1
\]

Therefore $2683 \div 3 = 894$ plus a remainder of 1.

This solution helps to illustrate why the method of long division always works! Let’s take a look at this exercise using that approach. Can you identify the similarities?
Rational Numbers (fractions) ($\mathbb{Q}$): The set of rational numbers is the set $\mathbb{Q}$ where numbers are of the following form:

$$\frac{a}{b} \quad \text{where } a \text{ and } b \text{ are integers and } b \neq 0.$$

Here, $a$ is called the numerator and $b$ is called the denominator. For example, $\frac{1}{3}$ is a rational number with a numerator of $a = 1$ and a denominator of $b = 3$.

Interpretations of Fractions

I. Fractions are division problems! i.e. $\frac{a}{b} = a \div b$. For example, $2683 \div 3 = \frac{2683}{3}$.

How many ways can we express the number 1 as a fraction? ____________

Using this interpretation, can you explain why we must omit $b = 0$ from the definition?

II. Fractions are symbols that represent a part-to-whole relationship. $\frac{a}{b}$ means that “$a$” objects are collected for each group of “$b$” objects. i.e. The numerator represents the part and the denominator represents the group that produces one whole. Below, the shaded regions in each example represent the same relationship: For each group of five objects, ____________ objects are collected.

III. Fractions can be interpreted numerically, where the quantity represents the part-to-whole relationship. e.g. $\frac{2}{5} = 0.4$ represents the relationship above.
Equivalent Fractions

**Rules**

**Definitions:**

- **Equivalent fractions:** We say that two fractions are equivalent when they represent the same part-to-whole relationship.

**How does this work?**

1. Fill in the shaded region so that the fractions are equivalent:

   ![Diagram](image)

   \( \frac{3}{4} = \frac{\square}{12} \)

   (a) **S1** Identify the number of groups/“wholes” present in the completed diagram, then compare to the incomplete diagram.

      In this example, we increase from one group of 4 (on the left side) to ___________ groups of 4 (on the right side).

   (b) **S2** Identify the number that makes one part (on the completed side), then calculate the number of parts needed (on the incomplete side) in order to maintain the part-to-whole relationship.

      Here, because we increased to three groups, we must shade three sections for each of these groups. i.e. shade 3 · ____ sections.

2. Suppose you get a sales job where you receive $3 for every five-dollar sale made.

   (a) If you make a total of six five-dollar sales, how much money was made in all?

   (b) How much of this money is your employer obligated to give you in return?

   (c) What are the equivalent fractions that represent the part-to-whole relationship of your pay?
(a) \( \frac{2}{5} = \frac{4}{10} = 0.4 \)

From the diagrams in (II) on page 4, we could see that these fractions were equivalent. This is also evident using fraction number lines.

(b) \( \frac{1}{3} = \frac{2}{6} = \frac{4}{12} \)

**Note:** The only difference in the diagrams below is the number of *groups* /“wholes” present. More specifically, the number of groups increases from one group of 3 to two groups of 3, and then to four groups of 3. Therefore, in each case, we simply continue to shade *one* rectangle for *each group of three* that we see.

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**Strategy!**

In general, we can create equivalent fractions by

- **Doubling** or **multiplying** the numerator and denominator by the **same** factor!
Improper and Mixed

**Definitions:**

1. **improper fraction:** We say that a fraction is improper when the numerator is greater than the denominator. For example, \( \frac{2683}{3} \) is an improper fraction and simply means: “2683 divided by 3” or that “we have a total of 2,683 parts whenever 3 parts produce one whole.”

2. **mixed number:** A mixed number has the form of an integer beside a fraction and represents a number separated into two parts: the total number of wholes, plus the remaining fractional part. For example, let’s express \( \frac{2683}{3} \) as a mixed number.

   **Solution:** From our long division on page 3 we found that \( 2683 \div 3 = 894 \text{ R}1 \). Therefore, there are 894 groups of 3 (i.e. 894 wholes), plus ”1 out of 3” remaining. Therefore, \( \frac{2683}{3} = 894\frac{1}{3} \).

**Making Connections**

1. To create a mixed number from an improper fraction, we simply need to divide (as show in the example above).

2. Let’s explain how to create an improper fraction from a mixed number.

   Well, an improper fraction simply tells us “how many parts we have in all (which is given by the numerator), whenever a whole consists of a certain number of parts (which is given by the denominator).” This understanding gives the following result:

   \[
   a \frac{b}{c} = \frac{(a \cdot c) + b}{c}
   \]

   **Why?** Because if \( a \) represents the number of wholes, and \( c \) represents the number of parts in each whole, then \( a \cdot c \) gives the total number of parts within “\( a \)” wholes. Then, we need to add the remaining \( b \) parts to get the total!
1. **Ratio**: A ratio is a pair of numbers $a$ and $b$, expressed in the form $a : b$ where $b \neq 0$, and can represent one of the following relationships: part-to-part relationships, part-to-whole relationships or whole-to-part relationships. Fractions are commonly used to represent ratios! In general, $a : b = \frac{a}{b}$.

For example, we can use ratios to compare the size of centimetres to metres: $100 : 1$. This reads, "one-hundred to one" since we get one-hundred centimetres for every metre. This is an example of a part-to-whole relationship, which produces the fraction: $\frac{100}{1}$.

2. **Equivalent Ratio**: Two ratios are said to be equivalent if the fractions that represent them are equivalent.

3. **Proportion**: A proportion is a statement that equates two ratios or fractions.

### Worked Example

Suppose you are making pancakes for yourself and two friends. If the recipe that you are using serves 12 people and calls for 4 eggs, how many eggs will you need to modify the recipe so that there are no leftovers?

**Solution**: We can use equivalent fractions or ratios. Using ratios: we know that the recipe’s ratio of people to eggs is given by $12 : 4$. If we are serving 3 people, we want to satisfy the following proportion: $12 : 4 = 3 : \text{______}$. The strategies are as follows:

1. Identify the side with the complete ratio.
2. Determine the operation needed (division or multiplication) to transform the number on that side to its corresponding number on the opposite side of the equal sign.
3. Perform the same operation on the remaining number to find the missing value on the opposite side of the equal sign.

\[
12 : 4 = 3 : \text{______} \\
\text{divide by 4!} \\
\]
1. Determine the place value of the underlined digit:
   (a) 567,104       (b) 40,300       (c) 39,560
   (d) 3,485,098     (e) 348,502,847   (f) 2,385,769,573

2. Use powers of 10 to express the numbers in question 1 in expanded form.

3. Perform long division and express the solutions as mixed numbers.
   (a) \( \frac{235}{4} \)       (b) \( \frac{4587}{6} \)       (c) \( \frac{85765}{7} \)       (d) \( \frac{117257}{8} \)

4. Determine an equivalent fraction for each part, Then use the graph paper (on the next page) to represent these equivalent fractions with diagrams and number lines:
   (a) \( \frac{2}{7} \)       (b) \( \frac{3}{5} \)       (c) \( \frac{5}{8} \)       (d) \( \frac{3}{10} \)

5. Determine the equivalent improper fraction for each mixed number:
   (a) \( 4\frac{2}{7} \)       (b) \( 9\frac{1}{8} \)       (c) \( 11\frac{2}{3} \)       (d) \( 4\frac{11}{12} \)

6. Determine which statements represent proportions. Justify your answer with an explanation:
   (a) \( \frac{2}{8} \div \frac{8}{32} \)       (b) \( \frac{3}{7} \div \frac{6}{21} \)
   (c) \( 18 : 9 \div 2 : 1 \)       (d) \( 24 : 6 \div 8 : 3 \)

7. Express ratios for the (a) part-to-part (b) part-to-whole and (c) whole-to-part relationships of the following:
   e.g. 4 dogs, 7 cats, and the total number of pets
   Solution:       (a) 4 : 7 (dogs to cats) or 7 : 4 (cats to dogs)
                   (b) 4 : 11 (dogs to pets) or 7 : 11 (cats to pets)
                   (c) 11 : 4 (pets to dogs) or 11 : 7 (pets to cats)

   i. 5 red marbles, 12 blue marbles, total number of marbles
   ii. 3 cups of green paint, 6 cups of purple paint, total amount of paint in the mixture
Let's Play!
Level: Q